Weighted Average

The centroid or center of mass of a planar region is the point at which that region balances perfectly, like a plate on the end of a stick. The coordinates of the centroid are given by weighted averages.

The x coordinate of the centroid is $\bar{x} = \dfrac{\int x\,dA}{\int dA}$, where dA is an infintessimal portion of area; the weighting function in this average is just x.

Similarly, the y coordinate of the centroid is $\bar{y} = \dfrac{\int y\,dA}{\int dA}$.

Find the centroid $(\bar{x}, \bar{y})$ of the parabolic region bounded by $x = -1$, $x = 3$, $y = (x-1)^2$ and $y = 4$.

1/9/25

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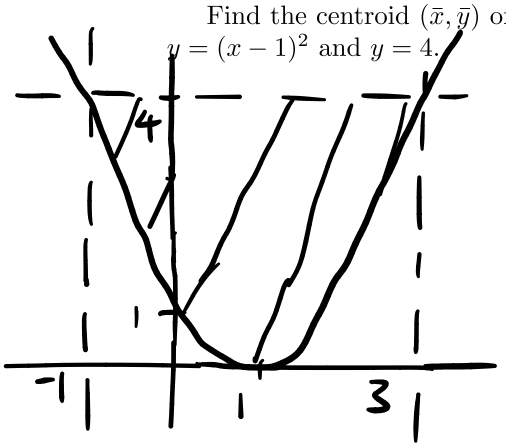

$y = (x-1)^2,\ y = 4$

$(x-1)^2 = 4$

$x - 1 = \pm 2$

$x = -1, 3$

$dA = (4 - y)\, dx$

$= 4 - (x-1)^2\, dx$

$= 4 - x^2 + 2x - 1\, dx$

$= -x^2 + 2x + 3\, dx$

$$\bar{x} = \frac{\int x\, dA}{\int dA}$$

$$= \frac{\int_{-1}^{3} x(-x^2+2x+3)\, dx}{\int_{-1}^{3} -x^2+2x+3\, dx}$$

$$= \frac{\int_{-1}^{3} x^3 - 2x^2 - 3x\, dx}{\int_{-1}^{3} x^2 - 2x - 3\, dx}$$

$$= \frac{\frac{x^4}{4} - \frac{2x^3}{3} - \frac{3x^2}{2}}{\frac{x^3}{3} - \frac{2x^2}{2} - 3x} \Bigg|_{-1}^{3}$$

$$\bar{x} = \frac{-\frac{45}{4} - \left(-\frac{7}{12}\right)}{-9 - \left(\frac{5}{3}\right)}$$

$$= \frac{\frac{135-7}{12}}{\frac{27+5}{3}}$$

$$= \frac{128}{4 \cdot 32}$$

$$= 1$$

$$\frac{1}{4} + \frac{2}{3} - \frac{3}{2}$$

$$= \frac{3 + 8 - 18}{12}$$

$$= -\frac{7}{12}$$

$$\bar{y} = \frac{\int y \, dA}{\int dA}$$

$$dA = \left((1+\sqrt{y}) - (1-\sqrt{y}) \right) dy$$

$$= 2\sqrt{y} \, dy$$

$$= \frac{\int_0^4 y \cdot 2\sqrt{y} \, dy}{\int_0^4 2\sqrt{y} \, dy}$$

$$\therefore (\bar{x}, \bar{y}) = \left(1, \frac{12}{5}\right)$$

$$= \left. \frac{2 \frac{y^{\frac{5}{2}}}{5/2}}{2 \frac{y^{\frac{3}{2}}}{3/2}} \right|_0^4$$

$$= \left. \frac{3}{5} y \right|_0^4$$

$$= \frac{12}{5}$$